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Effective Hamiltonians for strings and their spatial symmetry

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Abstract. The effective Hamiltonian governing the fluctuations of a $(d-n)$-dimensional string into the remaining n dimensions in a d bulk dimensional system is derived using semiclassical methods in the long-distance limit and found to represent the hypersurface area of the string. We further derive the effective Hamiltonian for strings with an associated O(2) Goldstone mode, and obtain a field theory defined on a curved surface given by the interface's position. The role of spatial symmetries in determining the form of the effective Hamiltonian is illustrated.

1. Introduction

Recently the interface between two phases in an Ising-like system has been much studied. In the context of field theoretic Landau–Ginsberg models, fluctuation corrections to the 'kink' ($\phi\alpha \tanh \lambda x$) mean-field theory interface profile have been studied by various authors (Rudnick and Jasnow 1978a, b, Ohta and Kawasaki 1977, see also Wallace and Zia 1979) to give results in reasonable agreement with experimental measurements on liquid gas systems (Huang and Webb 1968, Wu and Webb 1973).

Wallace and Zia (1979) have in addition used the 'surface tension' model for the interface in an Ising-like system (in which a $(d-1)$-dimensional interface fluctuates into the remaining dimension in a d bulk dimensional system with an energy proportional to its hypersurface 'area') in order to gain insight into the critical properties of the bulk system close to its lower critical dimension. This work was extended (Lowe and Wallace 1980) to consider the generalisation to the $(d-n)$-dimensional 'interface' or 'string' fluctuating into the remaining n dimensions in d bulk dimensions with a Hamiltonian, as before, proportional to the hypersurface area of the 'string'. It is hoped that insight is gained into the critical properties of systems possessing such strings.

In the context of a field theory these generalised surface tension models correspond to the effective (in the long-distance limit) Hamiltonian governing the interactions of the n Goldstone (gapless) modes which arise due to the spontaneous breaking of the spatial (Euclidean) symmetries by a solution of the Euler–Lagrange mean-field theory equations which depends on only n of the d coordinates of the system. Each one of the n Goldstone modes corresponds to a translation of the solution, and can be thought of as characterising the 'string' position. This is closely analogous to the O(N) nonlinear sigma model which arises when an O(N) symmetry is spontaneously broken.

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We shall review the derivation of the effective Hamiltonian governing the $n = 1, 2 \ldots$ string by semiclassical methods (Gervais *et al* 1976, Gervais and Sakita 1975). This will emerge in the long-distance limit which is equivalent to the 'zero-width' limit considered by other authors (Nielson and Oleson 1973, Förster 1974, Gervais and Sakita 1975). The derivation of the nonlinearly realised spatial symmetries and their role in determining the form of the effective Hamiltonian is considered.

We next consider the possibility of an extra O(2) Goldstone mode associated with a solution of the Euler–Lagrange equations in addition to the translation modes. The effective Hamiltonian, which emerges in the long-distance limit, is one which represents the nonlinear sigma model for the O(2) Goldstone mode on the curved surface given by the interface's position. The nonlinearly realised spatial symmetries are again considered.

The spatial symmetries of string-like objects (solitons) have also been studied in a related quantum system by Matsumoto *et al* (1981).

2. System with translation Goldstone modes only

In this section we shall consider interfaces or strings, specifically the interface in the ϕ^4 theory (Skryme 1962) and the vortices in the Abelian Higgs model (Abrikosov 1957, Marciano and Pagels 1978). In both these cases we have a localised solution of the Euler–Lagrange equations which have topological stability (a non-trivial winding number), and a finite energy density per unit hypersurface area. In the ϕ^4 model

$$H = \int \mathrm{d}^d \boldsymbol{x}[\tfrac{1}{2}(\nabla\phi(\boldsymbol{x}))^2 + \tfrac{1}{4}\lambda(\phi^2(\boldsymbol{x}) - m^2/\lambda)^2] \tag{2.1}$$

the relevant solution is

$$\phi(\boldsymbol{x}) = \frac{m}{\sqrt{\lambda}}\tanh\left(\frac{mx}{\sqrt{2}}\right) \tag{2.2}$$

which defines a $(d-1)$-dimensional surface which constitutes an interface between two regions of different phase. In the Abelian Higgs model (the Euclideanised version), where

$$H = \int \mathrm{d}^d x[+\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}|D_\mu\phi|^2 + \tfrac{1}{4}\lambda(|\phi|^2 - m^2/\lambda)^2] \tag{2.3}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad D_\mu\phi = (\partial_\mu - \mathrm{i}eA_\mu)\phi \tag{2.4}$$

and ϕ is a complex scalar field, the relevant solution has the following form

$$\phi(\boldsymbol{x}) = \phi_c(r)\,\mathrm{e}^{\mathrm{i}\omega} \qquad r = \sqrt{x_1^2 + x_2^2} \qquad \omega = \tan^{-1} x_2/x_1$$

$$A_\mu(\boldsymbol{x}) = A_c(r)\hat{\boldsymbol{\theta}} \qquad \hat{\boldsymbol{\theta}} = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.5}$$

which represents a $(d-2)$-dimensional 'string' in a d bulk dimensional system. The problem is to find a modified configuration which represents the 'string' or 'interface' fluctuating with a position relative to a flat reference hyperplane given by

$$x_a + f_a(y) = 0 \tag{2.6}$$

where y is the $d-n$ coordinates on the reference hyperplane and $a = 1, 2, \ldots, n$. The modified configuration must be a solution of the Euler–Lagrange equations to within errors which are ignorable in the long-distance limit (i.e. irrelevant in the renormalisation group sense).

The modifications to the solutions of the kind (2.2), (2.5) are as follows

$$x_a \rightarrow M_{ab}^{-1/2}(x_b + f_b(y)) \tag{2.7}$$

and for any vector with only its first n components non-zero

$$\begin{bmatrix} A \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \rightarrow \begin{bmatrix} M_{ab}^{-1/2} A_b \\ \\ \\ \\ \frac{\partial f_a}{\partial y_i} M_{ab}^{-1/2} A_b \end{bmatrix} \tag{2.8}$$

where

$$M_{ab} = \delta_{ab} + \frac{\partial f_a}{\partial y_i}\frac{\partial f_b}{\partial y_i} \tag{2.9}$$

and the inverse square root is taken of the matrix as a whole. Essentially this represents rotated versions of the original solutions (exactly if the f are linear in the y) (see Wallace 1980).

If we now insert this configuration into the original Hamiltonian and perform the x_a integrations we shall obtain the following effective Hamiltonian

$$H_{\text{eff}} \propto \int \mathrm{d}^{d-n}x \sqrt{\det g_{ij}} \tag{2.10}$$

where g_{ij} is the metric of the surface (2.6) and has the following form

$$g_{ij} = \delta_{ij} + \frac{\partial f_a}{\partial y_i}\frac{\partial f_a}{\partial y_j}. \tag{2.11}$$

In the $n = 1$ (ϕ^4) case H_{eff} reduces to the drumhead model derived by Diehl *et al* (1980). In analogy with the nonlinear sigma model the symmetries which are broken by the original solutions (2.2), (2.5) are nonlinearly realised on the fields $f_a(y)$ (see Wallace 1980).

In particular (small) rotations which mix one of the n axes perpendicular to the reference hyperplane with one of the $d-n$ axes in the reference hyperplane i.e.

$$x_a \rightarrow x_a + n_a\delta_i y_i \qquad y_i \rightarrow y_i - n_a x_a \delta_i \tag{2.12}$$

where n_a, δ_i are n, $(d-n)$-dimensional vectors and δ is small, can be shown to be realised to within acceptable (i.e. irrelevant) errors and, in the vortex case, to within a gauge transformation by the following change in the f fields

$$f_a(y) \rightarrow f_a(y) + \delta_i y_i n_a + n_b f_b \delta_i \frac{\partial}{\partial y_i} f_a(y), \tag{2.13}$$

on substitution into H_{eff} it can be straightforwardly shown that, up to surface terms, H_{eff} is unchanged.

3. Systems with additional Goldstone modes

It is possible for strings to possess other types of Goldstone modes than those which arise from the breaking of the spatial symmetry. Two examples will be studied in detail—an $n=1$ and $n=2$ string with an associated O(2) Goldstone mode. As before the formalism will suggest the generalisation to general n.

The $n=1$ case occurs in the following theory, studied by various authors (Lajzerowicz and Neiz 1979, Lawrie and Lowe 1981)

$$H=\int d^dx[\tfrac{1}{2}(\nabla S_1)^2+\tfrac{1}{2}|\nabla S_2|^2+\tfrac{1}{2}h|S_2|^2+\tfrac{1}{4}\lambda(S_1^2+|S_2|^2-r/\lambda)^2] \tag{3.1}$$

where S_1 is real and S_2 is complex. For $r<2h$ the stable interface with suitable boundary conditions on S_1 is the simple 'kink' in S_1 analogous to (2.1), with S_2 everywhere zero. For $r>2h$ the stable interface has the form

$$S_1=\sqrt{r/2}\tanh(\sqrt{h}x) \qquad S_2=(\sqrt{r-2h}/\lambda)\operatorname{sech}(\sqrt{h}x)\,e^{i\alpha} \tag{3.2}$$

where α is an arbitrary phase. Clearly this configuration breaks the O(2) symmetry in addition to the translation invariance of the system.

An $n=2$ string with an associated O(2) Goldstone mode occurs for the following modified Abelian Higgs model

$$H=\int d^dx[\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}|D_\mu\phi|^2+\tfrac{1}{4}\lambda(m_1^2/\lambda-|\phi|^2)^2+\tfrac{1}{2}|\partial_\mu\sigma|^2$$
$$+\beta(m_1^2/\lambda-|\phi|^2)|\sigma|^2+\tfrac{1}{2}h\sigma^2+\tfrac{1}{4}\gamma\sigma^4] \tag{3.3}$$

where σ is an additional complex scalar field. For sufficiently small h and large μ one obtains a stable string solution of the following form

$$\phi(x)=\tilde{\phi}_c(r)\,e^{i\omega} \qquad A(x)=\tilde{A}_c(r)\hat{\boldsymbol{\theta}} \qquad \sigma(x)=\sigma_c(r)\,e^{i\alpha} \tag{3.4}$$

where all the symbols are as in (2.5) but the functional form of $\tilde{A}_c$, $\tilde{\phi}_c$ will be different.

As before we wish to obtain the modified configurations in order to evaluate the effective Hamiltonians. The rules are the same as the previous section with the following additions

$$\alpha\rightarrow\theta(y)-\frac{\partial\theta(y)}{\partial y_i}\frac{\partial f_a}{\partial y_i}M_{ab}^{-1}(z_b+f_b) \qquad h\rightarrow h+\frac{\partial\theta(y)}{\partial y_i}g_{ij}^{-1}\frac{\partial\theta(y)}{\partial y_j}. \tag{3.5}$$

On performing the x_1, x_2 etc integrations after substituting the modified configurations into the Hamiltonians we obtain

$$H_{eff}\propto\int d^{d-n}y\,\sqrt{\det g_{ij}}\,G\Big(\frac{\partial\theta(y)}{\partial y_i}g_{ij}^{-1}\frac{\partial\theta(y)}{\partial y_j}\Big) \tag{3.6}$$

where the function G is of no universal significance. It will be determined by the details of the potentials etc in the origin Hamiltonian. The nonlinearly realised rotations of the kind (2.12) are realised as in (2.13) with the following additional change in $\theta(y)$:

$$\theta(y) \rightarrow \theta(y)+n_b f_b \delta_i \frac{\partial \theta(y)}{\partial y_i}. \tag{3.7}$$

Geometrically we have the picture of a field $\theta(y)$ defined on the surface given by the interface position, both in the interpretation of the Hamiltonian (3.6) and the non-linearly realised rotations (see Wallace 1980).

It can also be seen that H_{eff} will be invariant under the nonlinearly realised rotations whatever the form of the function G. The other features of the H_{eff} are consequences of the symmetries of the system and hence are of universal significance.

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